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THE EFFECT OF TEMPERATURE ON THE WIDTH OF A SMALL-AMPLITUDE, SOLITARY WAVE IN A COLLISION-FREE PLASMA

C. S. Gardner and G. K. Morikawa

March 15, 1961

AEC Research and Development Report

NEW YORK UNIVERSITY

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Magneto-Fluid Dynamics Division
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Physics and
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Abstract

Adlam and Allen [1] and Davis, Lüst, and Schlüter [4] have studied nonlinear plane-waves, propagating normal to the magnetic field, in a cold plasma. One solution of particular interest is a solitary wave, or single pulse. We present a method for solving the analogous problem for a plasma with finite temperature, in the limiting case where the amplitude of the wave is small -- and where, consequently, the width of the wave is very large. Our result is that in a warm plasma the solitary wave of small amplitude has the same form as in a cold plasma, but has a width which is larger than its width in a cold plasma by a factor which is approximately

$$\sqrt{1 + \frac{1}{16} \left(\frac{m_+}{m_-}\right)\beta} \quad .$$

Here β is the ratio of material pressure to magnetic pressure in the undisturbed plasma; and m_+, m_- are the respective masses of a positive ion and of an electron. Evidently this factor is appreciable, unless β is very small.

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The Effect of Temperature on the Width of a Small-Amplitude, Solitary Wave in a Collision-Free Plasma

1. Introduction.

Adlam and Allen [1] and Davis, Lüst, and Schlüter [4] have studied the nonlinear steady plane waves which can propagate normal to the magnetic field in a collision-free plasma. These authors assumed a cold plasma -- in which the temperature of the positive-ion gas is zero, and the temperature of the electron gas is zero. Thus at any point, in a cold plasma, all ions have the same velocity, and all electrons have the same velocity. A one-dimensional steady flow of a cold plasma is therefore governed by ordinary differential equation. These equations were solved by Adlam and Allen, and, in fuller detail, by Davis, Lüst, and Schlüter. In general the solution represents a periodic wave train (which is not sinusoidal, in this nonlinear theory). A particular solution, of special interest, represents a single pulse, or solitary wave. Gardner, et al. [5], Rose [14], Banos and Vernon [2] and Vernon [15] have obtained similar solutions from various two-fluid theories (based on suggestions due to Grad, [7] and [8]) which include pressure and temperature.

Our object is to investigate the effect of finite plasma temperature on the solitary wave solution. If no approximations are made, (other than that the plasma is collision-free), the equations of motion are too difficult to be solved for a warm plasma (with temperature). We are able, however, to find a solution in the limiting case of a wave of small amplitude. Our approximation here does not lead to a linear theory; a linear theory could never yield a solitary wave as solution.* The reason is that the width of the solitary wave tends to infinity as the amplitude tends to zero; so that the independent variable must be scaled accordingly. In fact, we assume the same scaling laws for the warm plasma as obtains for the solitary wave in the cold plasma. That is, we assume that the width of the wave varies inversely as the square root of its amplitude. This was shown for the cold plasma by Gardner and Morikawa [6]. Thus a length over which the magnetic and electric fields vary appreciably is very large in this limit. Consequently, one can obtain a formal solution of the collision-free Boltzmann equation by a successive-approximation calculation. This calculation yields expressions for the distribution functions of the ions and electrons in terms of the electric and magnetic field components and their derivatives. Inserting these expressions into Maxwell's equations, and retaining only the terms of lowest significant order, we obtain a differential equation for the asymptotic shape of the wave. This equation has the same form as the analogous equation for the cold plasma; however the coefficients depend on the plasma temperature. The final conclusion which can be drawn is that the effect of temperature is to multiply the width of the

---

* We point out that the inherent non-linearity (even for small amplitude) of the solitary wave solution in the study of gravity waves in water of finite depth has been known for many years (cf. Korteweg and de Vries, [10], and Keller, [9]). Other physical problems which exhibit this peculiarity are described more recently by Morikawa [12], [13].

Adlam-Allen solitary wave by a factor which is approximately

$$\sqrt{1 + \frac{1}{16} \left(\frac{m_+}{m_-}\right)\beta} \quad ,$$

(assuming for simplicity that the ion temperature and electron temperature are equal). Here β is the ratio of material pressure to magnetic pressure in the undisturbed plasma; and m_+ is the mass of a positive ion, and m_- the mass of an electron. Evidently this factor is appreciable unless β is very small. Another way of stating the result is that the cold-plasma treatment of the solitary wave is correct only if the electron sound speed is small compared to the Alfven speed in the undisturbed plasma. This result may be pertinent to the study of weak, collision-free plasma shocks (cf. Morawetz, [11]).

2. Equations of Motion. Dimensionless Variables and Scaling Transformations

Our problem is to study a plane wave, propagating with speed U_o in the negative $\bar{x}$ direction, in a plasma, initially in equilibrium, and permeated by a constant uniform field of strength B_o, in the $\bar{z}$ direction. We shall formulate this problem in a coordinate system which moves with the wave. We assume that in this coordinate system we have a steady-state flow of the plasma. Due to the coordinate transformation there is now a constant electric field, in the $\bar{y}$ direction, of magnitude $E_y = B_o U_o$. We use m.k.s. electrical units. Here, of course, B_o means the undisturbed magnetic field as it appears in the moving coordinate system. The plasma consists of positive ions and electrons continuously distributed. We assume that the motion of an ion or electron is determined only by the force exerted on it by the macroscopic electric and magnetic fields. The magnetic field is entirely in the $\bar{z}$ direction, and has strength $\bar{B}(\bar{x})$ (depending only on $\bar{x}$, and independent of time, in the frame of reference fixed to the wave). The electric field has the component $E_y = B_o U_o$ and also the component $E_x(\bar{x})$, due to charge separation. The plasma proper is described by the distribution functions of the ions and electrons, which are time-independent, and depend on only the one space coordinate $\bar{x}$.

Let $\bar{u}$, $\bar{v}$ denote the $\bar{x}$ and $\bar{y}$ components of the velocity of a particle. Let $\bar{f}_+(\bar{x},\bar{u},\bar{v})$ and $\bar{f}_-(\bar{x},\bar{u},\bar{v})$ denote the density of the ions and the density of the electrons in the $\bar{x}\ \bar{u}\ \bar{v}$ phase space. Then the distribution functions $\bar{f}_+$, $\bar{f}_-$ satisfy the following collisionless Boltzmann equations:

$$\bar{u}\,\frac{\partial \bar{f}_+}{\partial \bar{x}} + \frac{e}{m_+}\,(E_x + \bar{B}\bar{v})\,\frac{\partial \bar{f}_+}{\partial \bar{u}} + \frac{e}{m_+}\,(E_y - \bar{B}\bar{u})\,\frac{\partial \bar{f}_+}{\partial \bar{v}} = 0 \tag{1}$$

$$\bar{u}\,\frac{\partial \bar{f}_-}{\partial \bar{x}} - \frac{e}{m_-}\,(E_x + \bar{B}\bar{v})\,\frac{\partial \bar{f}_-}{\partial \bar{u}} - \frac{e}{m_-}\,(E_y - \bar{B}\bar{u})\,\frac{\partial \bar{f}_-}{\partial \bar{v}} = 0 \tag{2}$$

Here e is the electronic charge, and m_+ is the mass of an ion, and m_- the mass of an electron. The electric and magnetic fields are governed by the Maxwell equations,

$$(3) \qquad \frac{dE_x}{d\bar{x}} = \mu c^2 e \left[\iint \bar{f}_+ d\bar{u}d\bar{v} - \iint \bar{f}_- d\bar{u}d\bar{v}\right]$$

$$(4] \qquad \frac{d\bar{B}}{d\bar{x}} = -\mu e \left[\iint \bar{v}\bar{f}_+ d\bar{u}d\bar{v} - \iint \bar{v}\bar{f}_- d\bar{u}d\bar{v}\right]$$

where μ is the permeability of space and c the speed of light. (We use M.K.S. units.)

We are interested in solutions which tend to a constant state as $\bar{x}$ tends to $-\infty$. That is, we suppose the following conditions:

For $\bar{x} \rightarrow -\infty$,

$$(5) \qquad \bar{f}_+(\bar{x},\bar{u},\bar{v}) \rightarrow \bar{\psi}_+((\bar{u}-U_o)^2 + \bar{v}^2)$$

$$(6) \qquad \bar{f}_-(\bar{x},\bar{u},\bar{v}) \rightarrow \bar{\psi}_-((\bar{u}-U_o)^2 + \bar{v}^2)$$

$$(7) \qquad \bar{B}(\bar{x}) \rightarrow B_o \quad \text{(a constant)}$$

$$(8) \qquad E_x(\bar{x}) \rightarrow 0$$

where

$$(9) \qquad E_y = B_o U_o ,$$

and where the number densities of ions and electrons are equal:

$$\iint \bar{\psi}_+ d\bar{u}d\bar{v} = n_o = \iint \bar{\psi}_- d\bar{u}d\bar{v}$$

Here $\bar{\psi}_+$ and $\bar{\psi}_-$ are the undisturbed distribution functions for ions and for electrons. The most interesting case, of course, is the case wherein $\bar{\psi}_+, \bar{\psi}_-$ are Maxwellian with equal temperature (not a necessary restriction).

It is easily verified that one solution is given by

$$\bar{f}_+ \equiv \bar{\psi}_+, \quad \bar{f}_- \equiv \bar{\psi}_-, \quad \bar{B}(\bar{x}) \equiv B_o, \quad E_x \equiv 0.$$

This is the trivial solution -- corresponding to no disturbance of the plasma. It is not really clear than any other steady state solution does in fact exist. However, we shall assume that such a non-trivial solution exists, and furthermore that it depends asymptotically on a parameter ϵ in a certain way. We then proceed to find the solution for small ϵ, correct to lowest order in ϵ.

To define ϵ, we first must define a few other parameters of importance. Let p_+, p_- denote the initial pressures of the ion gas and of the electron gas; then

$$(10) \qquad p_+ = \frac{1}{2} m_+ \iint [(\bar{u}-U_o)^2 + \bar{v}^2]\bar{\psi}_+ d\bar{u}d\bar{v}$$

$$(11) \qquad p_- = \frac{1}{2} m_- \iint [(\bar{u}-U_o)^2 + \bar{v}^2]\bar{\psi}_- d\bar{u}d\bar{v} \quad .$$

Let a be the sound speed, and A the Alfven speed, of the undisturbed plasma; then

$$(12) \qquad a^2 = \frac{2[p_+ + p_-]}{n_o[m_+ + m_-]}$$

(13) $$A^2 = \frac{B_o^2}{\mu n_o[m_+ + m_-]} \quad .$$

Let the number M be defined by

(14) $$M^2 = \frac{U_o^2}{A^2 + a^2} \quad .$$

Then M is akin to Mach number, being the ratio of the flow speed U_o to the speed $(A^2 + a^2)^{1/2}$, which is the propagation speed of a small-amplitude hydromagnetic wave normal to the magnetic field. We define ϵ by setting

(15) $$M^2 = 1 + \epsilon^2 .$$

We define dimensionless variables (without bars) as follows:

(16) $$\bar{x} = \frac{1}{\epsilon} \frac{m_+ U_o}{eB_o} x$$

(17) $$\bar{u} = U_o(1 + u) \; ; \quad \bar{v} = U_o v$$

(18) $$\bar{B} = B_o(1 + \epsilon^2 B)$$

(19) $$E_x = \epsilon^3 B_o U_o E$$

(20) $$\bar{f}_+ = \frac{n_o}{U_o^2} f_+ \; ; \quad \bar{f}_- = \frac{n_o}{U_o^2} f_-$$

(21) $$\bar{\Psi}_+ = \frac{n_o}{U_o^2} \psi_+(u^2 + v^2); \quad \bar{\Psi}_- = \frac{n_o}{U_o^2} \psi_-(u^2 + v^2)$$

The reason for introducing the powers of ϵ into (16), (18),

(19) is that we assume that the length scale of the disturbance is of order $1/\epsilon$, and that the disturbance in B is of order ϵ^2, and the disturbance E_x is of order ϵ^3, as ϵ tends to zero. These scaling rules, which, as we shall see, are self-consistent, were motivated by their validity for the solitary wave in a cold plasma. The explicit solution is available for the solitary wave in a cold plasma. And, in agreement with the above, it is found that the non-trivial solution exists only if $M > 1$; and for $M^2 = 1 + \epsilon^2$, and E_x is of order ϵ^3, (as shown by Gardner and Morikawa).

In terms of the new variables given by (16)-(20), the Boltzmann equations take on the following forms:

$$\frac{\partial f_+}{\partial \theta} + \epsilon^2 B \frac{\partial f_+}{\partial \theta} = \epsilon(1+u) \frac{\partial f_+}{\partial x} - \epsilon^2 B \frac{\partial f_+}{\partial v} + \epsilon^3 E \frac{\partial f_+}{\partial u} \tag{22}$$

$$\frac{\partial f_-}{\partial \theta} + \epsilon^2 B \frac{\partial f_-}{\partial \theta} = -\epsilon\alpha(1+u) \frac{\partial f_-}{\partial x} - \epsilon^2 B \frac{\partial f_-}{\partial v} + \epsilon^3 E \frac{\partial f_-}{\partial u} \quad . \tag{23}$$

Here θ is a polar coordinate in the u,v plane; we define w, θ by

$$w = u^2 + v^2 \tag{24}$$

$$u = w^{1/2} \cos \theta \ , \quad v = w^{1/2} \sin \theta \tag{25}$$

so that

$$\frac{\partial}{\partial \theta} = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} \quad . \tag{26}$$

The coefficient α in (23) is the ratio of the masses of the two types of particle:

(27) $$\alpha = \frac{m_-}{m_+} .$$

The Maxwell equations now take on the following forms:

(28) $$\epsilon^4(1+\alpha)\,\frac{A^2}{c^2}\,\frac{dE}{dx} = [\iint f_+ du dv - \iint f_- du dv]$$

(29) $$\epsilon^3[-(1+\alpha) + 2(1+\epsilon^2)(P_+ + \alpha P_-)]\frac{dB}{dx} = (1+\epsilon^2)[\iint v f_+ du dv - \iint v f_- du dv].$$

Here P_+ and P_- are dimensionless pressures defined by

(30) $$P_+ = \frac{p_+}{n_o m_+ U_o^2} \quad , \quad P_- = \frac{p_-}{n_o m_- U_o^2} \quad .$$

Finally, we have the following conditions as x tends to $-\infty$:

For $x \longrightarrow -\infty$,

(31) $$f_+(x,u,v) \longrightarrow \psi_+(u^2 + v^2)$$

(32) $$f_-(x,u,v) \rightarrow \psi_-(u^2 + v^2)$$

(33) $$B(x) \longrightarrow 0$$

(34) $$E(x) \longrightarrow 0$$

and

(35) $$\iint \psi_+ du dv = 1, \qquad \iint \psi_- du dv = 1$$

$$
(36) \quad \begin{cases} \frac{1}{2} \iint (u^2 + v^2)\psi_+ du\, dv = P_+ \quad , \\ \frac{1}{2} \iint (u^2 + v^2)\psi_- du\, dv = P_- \quad . \end{cases}
$$

3. Series Solution of the Collision-free Boltzmann Equations.

Our procedure for solving the system (22), (23), (28), (29), with initial conditions (31)-(36) is to put

$$
f_+ = f_+^{(o)} + \epsilon f_+^{(1)} + \epsilon^2 f_+^{(2)} + \ldots
$$

$$
f_- = f_-^{(o)} + \epsilon f_-^{(1)} + \epsilon^2 f_-^{(2)} + \ldots
$$

in the Boltzmann equations (28), (29), and equate the coefficient of each power of ϵ on the left with the coefficient of the same power of ϵ on the right in the usual way. We thereby obtain a sequence of equations, which we can solve explicitly, and we obtain $f^{(o)}$, $f^{(1)}$, etc. in terms of B(x), E(x) and their derivatives. A different, but related scheme, is used by Chew, Goldberger and Low [3], the difference being in the scale transformations. We then insert our solution of the Boltzmann equation into the Maxwell equations (28), (29) and obtain differential equations -- to any order -- E(x), B(x). Retaining only the terms of lowest significant order, we finally arrive at an equation of third order for B(x). This equation can be solved by quadratures. The solution yields the asymptotic shape of the solitary wave in a warm plasma.

To carry out the iteration process for solving the Boltzmann equations, we can work with the ion equation (22) only, since the results for the electron equation can be obtained from the result for the ion equation by replacing

$$\frac{\partial}{\partial x} \quad \text{by} \quad -\alpha \frac{\partial}{\partial x} .$$

Thus we may consider the following:

$$(37) \qquad \frac{\partial f}{\partial \theta} + \epsilon^2 B(x) \frac{\partial f}{\partial \theta} = \epsilon(1+u) \frac{\partial f}{\partial x} - \epsilon^2 B(x) \frac{\partial f}{\partial v} + \epsilon^3 E(x) \frac{\partial f}{\partial u}$$

where

$$u = w^{1/2} \cos \theta, \qquad v = w^{1/2} \sin \theta$$

with the conditions for $x \longrightarrow -\infty$,

$$(38) \qquad f(x,u,v) \longrightarrow \psi(w)$$

$$(39) \qquad B(x) \rightarrow 0$$

$$(40) \qquad E(x) \rightarrow 0 .$$

We solve (37) by setting

$$(41) \qquad f = f_o + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \epsilon^4 f_4 + \ldots$$

and solving the sequence of equations, corresponding to the successive powers of ϵ, which we get on inserting (41) into (37). We use as initial conditions, to determine the constants of integration, the conditions

for $x \longrightarrow -\infty$,

$$(42) \qquad f_o \longrightarrow \psi, \qquad f_1 \longrightarrow 0, \qquad f_2 \longrightarrow 0 \quad \text{, etc.}$$

$$(43) \qquad B(x) \rightarrow 0, \qquad B'(x) \longrightarrow 0, \qquad B''(x) \longrightarrow 0, \quad \text{etc.}$$

$$(44) \qquad E(x) \rightarrow 0, \qquad E'(x) \longrightarrow 0, \qquad E''(x) \longrightarrow 0, \text{ etc.}$$

We shall have to compute f_o, f_1, f_2, f_3, and f_4 in order to obtain the lowest order approximation of the field variables B, etc.

In the lowest order, we have

$$\frac{\partial f_o}{\partial \theta} = 0$$

whence

$$f_o = \phi_o(x,w)$$

where ϕ_o is yet to be determined. The function ϕ_o will be determined by a compatibility condition, for the solvability of the next-order equation for f_1. In the next order, we have

$$(45) \qquad \frac{\partial f_1}{\partial \theta} = (1+u) \frac{\partial f_o}{\partial x} = (1+u) \frac{\partial}{\partial x} \phi_o(x,w) \ .$$

We integrate both sides with respect to θ, holding x,w fixed, from $\theta = 0$ to $\theta = 2\pi$. We get

$$(46) \qquad 0 = \frac{\partial \phi_0}{\partial x} .$$

Using the initial condition (42), we get

$$(47) \qquad \phi_0 = \psi(w); \quad \text{so that} \quad f_0 = \psi(w).$$

Now, from (45), (46) we have

$$(48) \qquad \frac{\partial f_1}{\partial \theta} = 0$$

whence

$$(49) \qquad f_1 = \phi_1(x,w).$$

Again, ϕ_1 will be determined from the next equation (coefficient of ϵ^2).

This next equation is

$$(50) \qquad \frac{\partial f_2}{\partial \theta} + B\frac{\partial f_0}{\partial \theta} = (1+u)\frac{\partial f_1}{\partial x} - B\frac{\partial f_0}{\partial v} .$$

Using (47), (49), this becomes

$$(51) \qquad \frac{\partial f_2}{\partial \theta} = (1+u)\frac{\partial}{\partial x}\phi_1(x,w) + B\psi'(w)[-2v].$$

Integrating with respect to θ, we get

$$(52) \qquad 0 = \frac{\partial}{\partial x}\phi_1(x,w).$$

Thus, by (42), we obtain

(53) $$\phi_1 = 0; \quad \text{so that} \quad f_1 = 0.$$

Now (51) becomes

$$\frac{\partial f_2}{\partial \theta} = B\psi'(w)[-2v]$$

whence

(54) $$f_2 = B\psi'(w)[2u] + \phi_2(x,w).$$

Again, to find ϕ_2 we go to the next equation (coefficient of ϵ^3).

The next equation is

(55) $$\frac{\partial f_3}{\partial \theta} + B\frac{\partial f_1}{\partial \theta} = (1+u)\frac{\partial f_2}{\partial x} - B\frac{\partial f_1}{\partial v} + E\frac{\partial f_o}{\partial u}$$

or, by (47), (53), (54),

(56) $$\frac{\partial f_3}{\partial \theta} = (1+u)\frac{\partial \phi_2}{\partial x} + B'\psi'(w)[2u + 2u^2] + E\psi'(w)[2u] .$$

Integrating (56) with respect to θ, we obtain

(57) $$0 = \frac{\partial \phi_2}{\partial x} + B'w\psi'(w)$$

whence

$$\phi_2 = - Bw\psi'(w)$$

and, by (54),

(58) $$f_2 = B\psi'(w)[-w + 2u].$$

Multiplying (57) by (1+u) and subtracting the result from (56), we get

$$(59) \qquad \frac{\partial f_3}{\partial \theta} = B'\psi'(w)[-w + (2-w)u + 2u^2] + E\psi'(w)[2u].$$

We can find an integral of this equation: We make use of the facts

$$\frac{\partial w}{\partial \theta} = 0 , \qquad \frac{\partial}{\partial \theta} = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} .$$

We find

$$(60) \qquad f_3 = B'\psi'(w)[(2-w)v + uv] + E\psi'(w)[2v] + \phi_3(x,w) .$$

Again, to find ϕ_3 we must use the next equation (coefficient of ϵ^4).

By now the nature of the algorithm should be clear. From this point on the calculations rapidly become lengthier. Therefore we shall not write out all the steps, but at this point merely give the results. We obtain the following:

$$(61) \qquad f_o = \psi$$

$$(62) \qquad f_1 = 0$$

$$(63) \qquad f_2 = B\psi'[-w + 2u]$$

$$(64) \qquad f_3 = B'\psi'[(2-w)v + uv] + E\psi'[2v]$$

$$(65) \qquad f_4 = B''\psi'[(\tfrac{7}{4} w - \tfrac{5}{8} w^2) + (-2+w)u + (-\tfrac{3}{2} + \tfrac{1}{2} w)u^2 - \tfrac{1}{3} u^3]$$
$$+ E'\psi'[\tfrac{3}{2} w - 2u - u^2] + B^2\psi'[(1+w)-4u]$$
$$+ B^2\psi''[\tfrac{1}{2} w^2 - 2wu + 2u^2] .$$

4. Computation of the Moments

We now compute the moments of f_+, f_- which occur in the Maxwell equations. We have

$$(66) \qquad \iint \psi(w)du\,dv = \pi \int_0^\infty \psi(w)dw = 1$$

and we define P, σ by

$$(67) \qquad 2P = \iint w\psi(w)du\,dv = \pi \int_0^\infty w\psi(w)dw$$

$$(68) \qquad 4\sigma P^2 = \iint w^2\psi(w)du\,dv = \pi \int_0^\infty w^2\psi(w)dw.$$

(Note that (68) is a fourth moment.) By applying Schwarz's inequality ($|\int F\cdot G|^2 \leq |\int F^2|\cdot|\int G^2|$) it can be shown that if ψ is positive, then

$$\sigma \geq 1$$

and that if $\psi(w)$ is positive and a monotone decreasing function of w, that

$$\sigma \geq 4/3.$$

If ψ is a Maxwellian distribution, then

$$\sigma = 2.$$

By integrating (61) to (65) we obtain the following:

$$(69) \qquad \iint f_0 du\,dv = 1$$

(70) $$\iint f_1 du\, dv = 0$$

(71) $$\iint f_2 du\, dv = B$$

(72) $$\iint f_3 du\, dv = 0$$

(73) $$\iint f_4 du\, dv = B''[-1 + \frac{3}{2} P] - E' + 2B^2.$$

Also, we obtain the following:

(74) $$\iint v f_o du\, dv = 0$$

(75) $$\iint v f_1 du\, dv = 0$$

(76) $$\iint v f_2 du\, dv = 0$$

(77) $$\iint v f_3 du\, dv = B'[-1 + 2P]$$

(78) $$\iint v f_4 du\, dv = 0.$$

We shall also need the moment of f_5. This can be obtained, without actually computing f_5, as follows: we multiply the Boltzmann equation (37) by u and integrate, noting that

$$\iint u \frac{\partial f}{\partial \theta} du\, dv = \iint u[u \frac{\partial f}{\partial v} - v \frac{\partial f}{\partial u}] du\, dv$$

$$= \iint [\frac{\partial}{\partial v} (u^2 f) - \frac{\partial}{\partial u} (uvf)] du\, dv + \iint vf\, du\, dv$$

$$= \iint vf\, du\, dv.$$

We obtain, then,

$$(1+\epsilon^2 B) \iint vf\, du\, dv = \epsilon \frac{\partial}{\partial x} \iint (u+u^2) f\, du\, dv - \epsilon^3 E \iint f\, du\, dv$$

so that

$$\iint v\, f_5 du\, dv + B \iint v f_3 du\, dv = \frac{\partial}{\partial x} \iint (u+u^2) f_4 du\, dv - E \iint f_2 du\, dv.$$

In this way we obtain

$$\iint v f_5 du\, dv = B'''[1 - \tfrac{11}{4} P + \tfrac{3}{2} \sigma P^2] + E''[1 - \tfrac{3}{2} P] + 3BB'. \tag{79}$$

From these results we obtain the following:

$$\iint f_+ du\, dv = 1 + \epsilon^2 B + \epsilon^4 \left\{ B''[-1 + \tfrac{3}{2} P_+] - E' + 2B^2 \right\} + 0(\epsilon^6) \tag{80}$$

$$\iint f_- du\, dv = 1 + \epsilon^2 B + \epsilon^4 \left\{ \alpha^2 B''[-1 + \tfrac{3}{2} P_-] + \alpha E' + 2B^2 \right\} + 0(\epsilon^6) \tag{81}$$

$$\begin{aligned} \iint v f_+ du\, dv = {} & \epsilon^3 \left\{ B'[-1 + 2P_+] - E \right\} \\ & + \epsilon^5 \left\{ B'''[1 - \tfrac{11}{4} P_+ + \tfrac{3}{2} \sigma_+ P_+^2] + E''[1 - \tfrac{3}{2} P_+] + 3BB' \right\} + 0(\epsilon^7) \end{aligned} \tag{82}$$

$$\begin{aligned} \iint v f_- du\, dv = {} & \epsilon^3 \left\{ B'[\alpha - 2\alpha P_-] - E \right\} \\ & + \epsilon^5 \left\{ B'''[-\alpha^3 + \tfrac{11}{4} \alpha^3 P_- - \tfrac{3}{2} \alpha^3 \sigma_- P_-^2] + E''[\alpha^2 - \tfrac{3}{2} \alpha^2 P_-] \right. \\ & \left. - 3\alpha BB' \right\} + 0(\epsilon^7) . \end{aligned} \tag{83}$$

These moments are sufficient for our purpose in the following sections. However, an appreciable amount of additional calculations will be necessary in order to obtain an explicit expression for the anisotropic pressure tensor corresponding to our distribution function. Then a more direct comparison of our results with other two-fluid theories will be possible.

5. Differential Equation for B

If we insert the expressions (80), (81), (82), (83) into the Maxwell equations (28), (29) we obtain the following:

$$(84)\qquad (1+\alpha)\frac{A^2}{c^2}E' = B''[-(1-\alpha^2) + \frac{3}{2}(P_+-\alpha^2 P_-]-(1+\alpha)E' + O(\epsilon^2)$$

$$\begin{aligned}(85)\qquad & [-(1+\alpha) + (1+\epsilon^2)(2P_+ + 2\alpha P_-)]B' \\ & = (1+\epsilon^2)\left\{B'[-(1+\alpha) + (2P_+ + 2\alpha P_-)]\right\} \\ & + \epsilon^2\left\{B'''[(1+\alpha^3)- \frac{11}{4}(P_+ + \alpha^2 P_-)+ \frac{3}{2}(\sigma_+ P_+^2 + \alpha^3 P_-^2)]\right. \\ & \left. + E''[(1-\alpha^2)- \frac{3}{2}(P_+ - \alpha^2 P_-)] + 3(1+\alpha)BB'\right\} + O(\epsilon^4).\end{aligned}$$

Since the lowest-order terms cancel, (85) can be written as follows:

$$\begin{aligned}(86)\qquad & B'''[(1+\alpha^3)- \frac{11}{4}[P_+ + \alpha^3 P_-) + \frac{3}{2}(\sigma_+ P_+^2 + \alpha^3 \sigma_- P_-^2)] \\ & + E''[(1-\alpha^2)- \frac{3}{2}(P_+-\alpha^2 P_-)] + 3(1+\alpha)BB' - (1+\alpha)B' = O(\epsilon^2).\end{aligned}$$

Now we eliminate E from (86) by use of (84). In this way we find a differential equation for B(x) of the following form (neglecting terms which are $O(\epsilon^2)$).

$$(87) \qquad RB''' + 3BB' - B' = 0.$$

The coefficient R is quite lengthy but can be simplified somewhat if the term containing A^2/c^2 in (84) is neglected. Neglecting this term is equivalent to assuming charge neutrality, instead of using Poisson's equation. With this simplification the coefficient R becomes

$$(88) \qquad R = \alpha + \frac{(1-12\alpha)}{4(1+\alpha)} (P_+ - \alpha^2 P_-)$$
$$+ \frac{1}{4(1+\alpha)^2} \left\{ 6(1+\alpha)(\sigma_+ P_+^2 + \alpha^3 \sigma_- P_-^2) - 9(P_+ - \alpha^2 P_-)^2 \right\} .$$

It is convenient to rewrite (88) in terms of β_+, β_-, which are defined as the ratios of the pressures of the ion gas and electron gas, respectively, to the magnetic pressure. That is,

$$(89) \qquad \beta_+ = \frac{p_+}{B_o^2/2\mu} \quad , \qquad \beta_- = \frac{p_-}{B_o^2/2\mu} \quad .$$

We have also

$$(90) \qquad P_+ = \frac{(1+\alpha)\beta_+}{2(1+\epsilon^2)(1+\beta_+ + \beta_-)} \quad , \quad \alpha P_- = \frac{(1+\alpha)\beta_-}{2(1+\epsilon^2)(1+\beta_+ + \beta_-)} \quad .$$

Inserting these expressions into (88), we obtain

$$(91) \qquad R = \alpha + \frac{(1-12\alpha)(\beta_+ - \alpha\beta_-)}{8(1+\beta_+ + \beta_-)}$$
$$+ \frac{1}{16} \left\{ 6(1+\alpha)(\sigma_+ + \beta_+^2 + \alpha\sigma_-\beta_-^2) - 9(\beta_+ - \alpha\beta_-)^2 \right\} .$$

We may assume

$$\alpha \ll 1$$

since the mass of an electron is much less than the mass of any positive ion. If we also assume that the ion and electron temperatures are equal, so that the pressures are equal, we have

$$\beta_+ = \beta_- = \frac{1}{2}\beta \quad .$$

For small β, which is an interesting case, (91) reduces to

(92) $$R = \alpha + \frac{\beta}{16} \quad .$$

Comparing this with the expression we get if we do not neglect A^2/c^2, we see that a necessary condition for the validity of the charge-neutral assumption is

(93) $$\frac{A^2}{c^2} \ll (\alpha + \frac{\beta}{16}).$$

6. Width of the Solitary Wave

The differential equation (87) can be solved by quadratures. The solution satisfying the condition

$$\text{for } x \rightarrow -\infty \,, \qquad B \rightarrow 0$$

is

$$B(x) = \frac{1}{\cosh^2\left[\frac{x}{2R^{1/2}}\right]} \tag{94}$$

(apart from a constant which may be added to x; this, however, may be removed by a shift of coordinates).

Plotted against x, this is a symmetric curve, rising from $B = 0$ at $x = -\infty$ to its maximum $B = 1$ at $x = 0$, and then again falling to $B = 0$ at $x = +\infty$. And B takes on half its maximum value at the two points where

$$\cosh\left(\frac{1}{2R^{1/2}}\right) = 2^{1/2}$$

that is, at the two points

$$x = \pm\, 2R^{1/2} \times 0.881 \quad .$$

Now by (14), (15), (16) defining the dimensionless variables x, we obtain from this the following expression for the width L of the solitary wave:

$$L \approx \frac{3.52}{\sqrt{M^2 - 1}} \sqrt{1 + \frac{1}{16}\left(\frac{m_+}{m_-}\right)\beta}\; \frac{\sqrt{m_- m_+}\, U_o}{eB_o}$$

which should be accurate for small (positive) values of $(M^2 - 1)$, and small β, if the condition (93) for the validity of the charge-neutral assumption obtains. Of course, under these assumptions U_o is nearly the Alfven speed, so we have the following alternative for L:

$$L \approx \frac{3.52}{\sqrt{M^2 - 1}} \sqrt{1 + \frac{1}{16} \left(\frac{m_+}{m_-}\right)\beta} \quad \frac{1}{e} \sqrt{\frac{m_-}{\mu n_o}} \quad .$$

The analogous formula for the cold plasma -- which can also be obtained directly (our earlier work) -- is found by setting $\beta = 0$.

Thus we see that the effect of plasma temperature on the Adlam-Allen solitary wave is to multiply the width of the wave by the factor

$$\sqrt{1 + \frac{1}{16} \left(\frac{m_+}{m_-}\right)\beta} \quad ,$$

for small β.

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